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Exact solutions of quantum mappings from the lattice KdV as multi-dimensional operator difference equations

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Abstract
The lattice KdV equation is defined on an associative algebra and a new initial value problem on the lattice KdV is considered. This leads to two new families of iterative mappings (discrete equations of motion) which can be solved for all (discrete) times. Imposing a certain commutation relation on the algebra makes it possible to prove (using a Yang–Baxter structure) that these mappings are quantum integrable maps. In the classical limit the maps become a new class of integrable mappings in the sense of Liouville–Arnold–Veselov (LAV).

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1. Introduction

The lattice KdV equation [26] already has quite a history (see [23] for a review). By performing a particular continuum limit, the full hierarchy of the KdV and higher-order KdV equations can be derived, as was shown in [36]. In [30] and [12] a periodic initial value problem was imposed on the lattice KdV; this leads to finite-dimensional reductions that are integrable rational mappings. A discrete analogue of the Painlevé II equation was derived from the lattice KdV in [25], see also [18]. In [22, 11] the rational mappings arising from a periodic initial value problem on the lattice KdV were quantized in the sense that the variables were replaced by their noncommuting analogues via the usual Dirac canonical quantization; the integrability of these 'quantum' mappings was proven using an associated quantum Yang–Baxter structure. In [29] it was shown that the lattice KdV can be used as a convergence acceleration algorithm.

In this paper a new initial value problem is imposed on the lattice KdV equation: a finite (or open-ended) 'staircase', as opposed to the periodic initial value problem of [30] and [12]. The quantum mappings (i.e., the mappings in terms of noncommuting variables) are derived

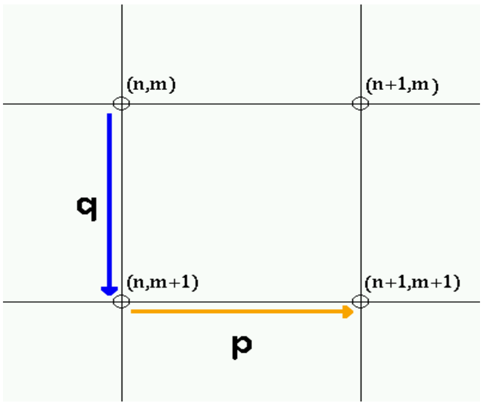

Figure 1. Elementary plaquette.

first, giving a new family of multi-dimensional quantum mappings. Taking the classical limit, a new family of multi-dimensional (classical) mappings is obtained. In both the quantum and classical cases, the discrete equations of motion (the maps themselves) can be solved explicitly for all time. (In this sense, the new mappings that arise from a finite staircase are simpler than those that arise from a periodic initial value problem. Even in the classical case, the solution of the mappings from a periodic initial value problem is still an object of research [24, 20].)

2. The lattice KdV equation

The lattice KdV partial *difference* equation relates the values of a field variable $u_{n,m}$ around an elementary plaquette (a quadrilateral) as follows,

$$((p-q)I + u_{n,m+1} - u_{n+1,m})((p+q)I + u_{n,m} - u_{n+1,m+1}) = (p^2 - q^2)I \tag{2.1}$$

$n, m \in \mathbb{Z}$ and where (n, m) can be taken as the coordinates of the field variable $u_{n,m}$ as depicted in figure 1. (Equation (2.1) is more accurately termed the 'lattice potential KdV'; however, we shall continue to refer to it as the lattice KdV as this is the usual name in the literature, see [23].) The lattice parameters $p, q \in \mathbb{C}$, although it will be assumed later, for the purposes of quantization, that $p^2 - q^2 \in \mathbb{R}$. In a somewhat imprecise sense, the lattice parameters p and q may be considered as the 'separation' of the lattice sites, as indicated in figure 1. In this paper, the field variables $u_{n,m} \in \mathcal{A}$, where $\mathcal{A}$ is an arbitrary associative algebra with unit I. The only additional structure assumed of $\mathcal{A}$ at this stage is the invertibility of the terms $((p+q)I + u_{n,m} - u_{n+1,m+1})$, for all $n, m \in \mathbb{Z}$. For convenience, I will not be written explicitly in expressions such as pI (i.e., in this case we would simply write p). (The lattice KdV, (2.1), still obeys the property of higher-dimensional consistency—the CAC property [27]—in this associative algebra case. The higher-dimensional consistency of some other integrable lattice equations in the associative algebra case was recently investigated in [8].)

It is easily seen that, on knowing the value of the field variable at three of the four sites around an elementary plaquette, the lattice KdV equation (2.1) gives the value of the field variable at the fourth site. This allows for the consideration of initial value problems on the lattice. Periodic initial value problems were first considered in [30], where the lattice KdV (2.1) was one of the partial difference equations considered (along with the mixed lattice MKdV-Toda equation). This led to finite-dimensional reductions of the lattice, i.e.,

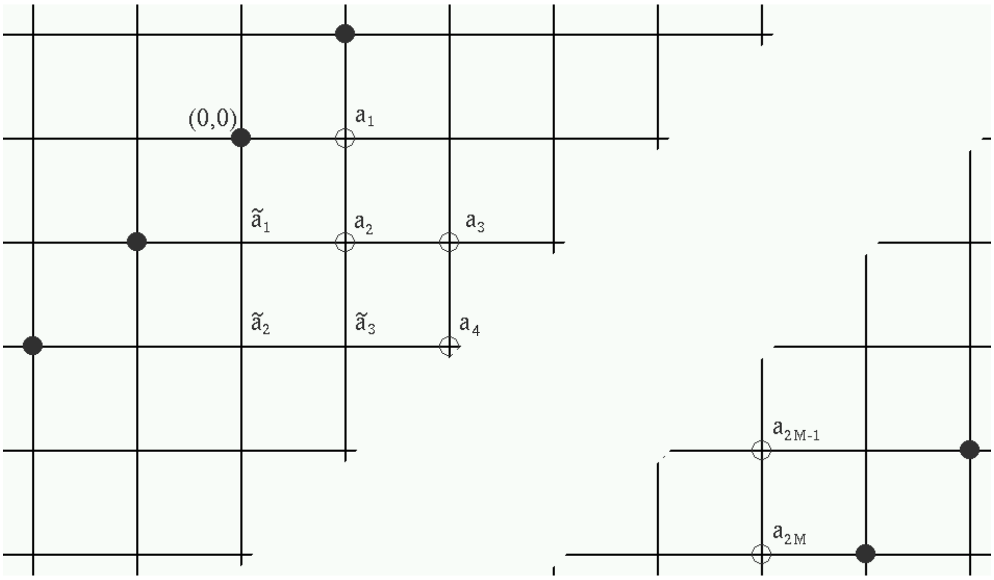

Figure 2. Evolution starting from an initial value staircase sandwiched between two diagonals consisting of infinity-valued field variables. The infinity-valued field variables are denoted by the larger black dots.

multi-dimensional families of mappings. These mappings turned out to be integrable [12]. In this paper a 'finite staircase' initial value problem is considered. This term is explained in the next section.

3. Finite staircase initial value problem

Previously, integrable mappings (in both the classical and quantum context) have been derived by considering 'staircase' or 'sawtooth' initial value problems on integrable lattices that were either periodic (such as in [15, 34, 14, 30, 12]) or quasiperiodic (such as in [15, 34, 13]). In this paper we give the first consideration of 'finite staircase' (or open-ended) initial value problems on the lattice. As in the above references, the lattice initial conditions consist of a staircase or 'sawtooth' pattern. What is new here is that the evolution of interest is considered to occur between two diagonals of infinity-valued field variables, as shown in figure 2. Infinity-valued field variables are given at sites for which $n + m = 0$, and at the parallel diagonal occurring immediately after the staircase of initial values. The initial data for the field variables are given along the staircase, as indicated in figure 2, as

$$a_{2j} := u_{j,j} \qquad a_{2j+1} := u_{j+1,j} \qquad j \in \mathbb{N}.$$

As specification of the value of the field variable at three points around an elementary plaquette gives the value of the field variable at the fourth point via the lattice KdV equation (2.1), it is easily seen how the 'evolution' proceeds. Points diagonally below and to the left are considered as being temporally 'later', and the 'time update' is denoted by an over-tilde. Also note that the backwards temporal evolution is given explicitly via the lattice KdV equation. And, further to this, it would be possible to consider an evolution outside the infinity-valued field variables; however, this evolution will not be considered here.

With regards to this type of initial condition for a discrete-time system, a precursor can be found in the 'open lattice' boundary condition applied to the (classical) discrete-time Toda lattice, as given by [31] (where LAV integrability was also proven). With regards to the lattice KdV, fixed columns of infinity-valued field variables were imposed in an application of the lattice KdV to convergence acceleration algorithms [29]. Nevertheless, the type of initial conditions on a lattice equation as given here, and the considerations, are new.

We consider two canonical cases of initial value staircase (those with an even number of initial value points and those with an odd number). All other evolutions along the lattice between two diagonals of infinity-valued field variables can be re-expressed in the form shown in figure 2 by a change of parameters and/or an evolution.

In the next two sections the evolutions that follow from the style of initial value problem shown in figure 2 will be explicitly considered, leading to multi-dimensional mappings, which will later be shown to be integrable.

3.1. The even staircase

Consider a staircase like that shown in figure 2 and consisting of an even number of points. Let the number of points $N = 2M$, where $M \in \mathbb{N}, M > 1$. The lattice KdV equation (2.1) may be rewritten in the form

$$u_{n,m+1} = u_{n+1,m} + q - p + (p^2 - q^2)(p + q + u_{n,m} - u_{n+1,m+1})^{-1}. \tag{3.1}$$

Throughout this paper we proceed in a formal sense, and, specifically, we assume the invertibility of $(p + q + u_{n,m} - u_{n+1,m+1})$. As of yet, no mention has been made of the commutativity, or otherwise, of the field variables. For the moment we still refrain from specifying commutation relations (no commuting will occur).

Equation (3.1) and figure 2 lead to

$$\begin{aligned} \tilde{a}_1 &= a_1 + q - p \\ \tilde{a}_{2j+1} &= a_{2j+1} + q - p + (p^2 - q^2)(p + q + a_{2j} - a_{2j+2})^{-1} \\ \tilde{a}_{2j} &= a_{2j} + q - p + (p^2 - q^2)(p + q + \tilde{a}_{2j-1} - \tilde{a}_{2j+1})^{-1} \\ \tilde{a}_N &= a_N + q - p \end{aligned} \tag{3.2}$$

for $1 \leqslant j < M$. The first of these equations is obtained by setting $a_0 \equiv u_{0,0}$ to $a_0 = cI$, where $c \in \mathbb{C}$, and taking the limit $c \to \infty$. (This is straightfoward in a finite-dimensional matrix representation.) The last equation is obtained similarly. Noting that the natural variables for the mapping appear to be the differences of adjacent even-numbered or adjacent odd-numbered field variables, define (as in [30, 12, 22, 11])

$$v_k := p + q + a_k - a_{k+2} \qquad k \in \mathbb{Z} \tag{3.3}$$

(for this mapping we are interested in $1 \leqslant k \leqslant N - 2$). Define the constant

$$a := q^2 - p^2.$$

Therefore the mapping reads

$$\begin{aligned} \tilde{v}_1 &= v_1 + a v_2^{-1} \\ \tilde{v}_{2i+1} &= v_{2i+1} - a v_{2i}^{-1} + a v_{2i+2}^{-1} \\ \tilde{v}_{2i} &= v_{2i} - a \tilde{v}_{2i-1}^{-1} + a \tilde{v}_{2i+1}^{-1} \\ \tilde{v}_{N-2} &= v_{N-2} - a \tilde{v}_{N-3}^{-1} \end{aligned} \tag{3.4}$$

for $1 \leqslant i < M - 1$. At this stage (before the specification of commutation relations) a finite-dimensional representation of $\mathcal{A}$ could be considered where the $\{v_i\}$ are invertible matrices of some finite dimension. The classical (i.e., commuting algebra $\mathcal{A} \equiv \mathbb{C}$) case with $M = 2$ was given previously in [35].

We will say that the mapping (3.4) has $l_E := M - 1$ 'degrees-of-freedom'. However, it is not strictly correct to number the degrees-of-freedom when the map is defined on $\mathcal{A}$ without specifying appropriate commutation relations. Hence $l_E := M - 1$, strictly speaking, only labels the length of the initial value staircase that the mapping is derived from. Later, in section 7, commutation relations are specified, making it possible to count the actual number of degrees-of-freedom.

It is evident from the definitions that knowing the value of a_1 along with the values of all $\{v_{2i-1}\}$ allows for the complete reconstruction of the values of the $\{a_{2i-1}\}$. From equation (3.2) it is seen that a_1 evolves as $\tilde{a}_1 = a_1 + q - p, \tilde{\tilde{a}}_1 = a_1 + 2(q - p), \ldots$. Therefore, also evolving with the map (3.4) allows the reconstruction of the values of the odd-numbered a_i at any time-level. Similarly, knowing the value of a_N along with the values of all of the $\{v_{2i}\}$ allows the reconstruction of the even-numbered a_i.

3.2. The odd staircase

Consider a staircase like that shown in figure 2 and consisting of an odd number of points. Let the number of points $N = 2M + 1$, where $M \in \mathbb{N}, M > 1$.

Equation (3.1) and figure 2 lead to

$$\begin{aligned}
\tilde{a}_1 &= a_1 + q - p \\
\tilde{a}_{2i+1} &= a_{2i+1} + q - p + (p^2 - q^2)(p + q + a_{2i} - a_{2i+2})^{-1} \\
\tilde{a}_N &= a_N + q - p \\
\tilde{a}_{2j} &= a_{2j} + q - p + (p^2 - q^2)(p + q + \tilde{a}_{2j-1} - \tilde{a}_{2j+1})^{-1}
\end{aligned} \tag{3.5}$$

for $1 \leqslant i < M, 1 \leqslant j \leqslant M$. Hence, similarly to section 3.1, the mapping reads

$$\begin{aligned}
\tilde{v}_1 &= v_1 + a v_2^{-1} \\
\tilde{v}_{2i+1} &= v_{2i+1} - a v_{2i}^{-1} + a v_{2i+2}^{-1} \\
\tilde{v}_{N-2} &= v_{N-2} - a \tilde{v}_{N-3}^{-1} \\
\tilde{v}_{2j} &= v_{2j} - a \tilde{v}_{2j-1}^{-1} + a \tilde{v}_{2j+1}^{-1}
\end{aligned} \tag{3.6}$$

for $1 \leqslant i < M - 1, 1 \leqslant j \leqslant M - 1$.

It is easily seen that

$$\sum_{j=1}^{M} v_{2j-1} =: \nu \tag{3.7}$$

is a constant of the mapping (i.e., it is an invariant of the time-evolution given by the mapping). Later, in section 7, after specifying appropriate commutation relations, it will be seen to be a Casimir.

We will say that the 'odd' mapping, (3.6), has $l_O := M - 1$ 'degrees-of-freedom'. However, note again that 'degrees-of-freedom' does not strictly apply until appropriate commutation relations are specified.

Again, knowing the value of one odd and one even a_i allows for the reconstruction of the original lattice variables; however, this time, things are not quite so simple with regards to the reconstruction at later 'times'. This is, of course, due to the lack of a linearly evolving a_{2i} in this case.

4. Lax matrices

In this section, Lax matrices are derived for the mappings given in the previous section. The Lax matrices give the invariants of the mapping.

The Lax matrices are derived using the Lax matrices for the lattice KdV equation. (The Lax matrices continue to hold in the associative algebra case; the lattice KdV defined on $\mathcal{A}$, (2.1), follows from the Lax pair via a straightforward calculation. Alternatively, the Lax matrices can be *derived* from equation (2.1) by the algorithm based on the CAC property that was given in [9] and [21].) The Lax pair for the lattice KdV consists of an $\mathfrak{L}$-part, which effects a 'horizontal' movement in the associated linear problem,

$$\mathfrak{L}_{(n,m)} := \mathfrak{U}_{(n,m)}\mathfrak{P}\mathfrak{U}^{-1}_{(n+1,m)}$$

where

$$\mathfrak{U}_{(n,m)} = \begin{pmatrix} 1 & 0 \\ u_{n,m} & 1 \end{pmatrix} \qquad \mathfrak{P} = \begin{pmatrix} p & 1 \\ k^2 & p \end{pmatrix}$$

and an $\mathfrak{M}$-part, which effects a 'vertical' movement,

$$\mathfrak{M}_{(n,m)} := \mathfrak{U}_{(n,m)}\mathfrak{Q}\mathfrak{U}^{-1}_{(n,m+1)}$$

where

$$\mathfrak{Q} = \begin{pmatrix} q & 1 \\ k^2 & q \end{pmatrix}.$$

The associated linear problem is as follows:

$$\mathfrak{L}_{(n,m)}\Phi_{n,m} = (p-k)\Phi_{n+1,m} \qquad \mathfrak{M}_{(n,m)}\Phi_{n,m} = (q-k)\Phi_{n,m+1}.$$

We first consider the even case, as it turns out that the odd case is, for the most part, the same as the even case but with a slight complication at the end of the initial value staircase to account for the last lattice site. Suppose that $2 \leqslant m \leqslant M-2$ and consider the compatibility of the paths shown in figure 3. The anticlockwise path $a_{2m-1} \to a_{2m} \to \tilde{a}_{2m-1} \to \tilde{a}_{2m} \to \tilde{a}_{2m+1}$ is effected by

$$\widetilde{\mathfrak{L}}_{2m}\widetilde{\mathfrak{M}}_{2m-1}\widetilde{\mathfrak{L}}^{-1}_{2m-1}\mathfrak{M}_{2m-1}.$$

The clockwise path $a_{2m-1} \to a_{2m} \to a_{2m+1} \to a_{2m+2} \to \tilde{a}_{2m+1}$ is effected by

$$\widetilde{\mathfrak{L}}^{-1}_{2m+1}\mathfrak{M}_{2m+1}\mathfrak{L}_{2m}\mathfrak{M}_{2m-1}.$$

Equating the anticlockwise and clockwise paths gives

$$\widetilde{\mathfrak{A}}_{2m}\mathfrak{P}\widetilde{\mathfrak{A}}^{-1}_{2m+1}\widetilde{\mathfrak{A}}_{2m-1}\mathfrak{Q}\widetilde{\mathfrak{A}}^{-1}_{2m}\mathfrak{A}_{2m}\mathfrak{P}^{-1}\widetilde{\mathfrak{A}}^{-1}_{2m-1}\mathfrak{A}_{2m-1}\mathfrak{Q}\mathfrak{A}^{-1}_{2m}$$
$$= \mathfrak{A}_{2m+2}\mathfrak{P}^{-1}\widetilde{\mathfrak{A}}^{-1}_{2m+1}\mathfrak{A}_{2m+1}\mathfrak{Q}\mathfrak{A}^{-1}_{2m+2}\mathfrak{A}_{2m}\mathfrak{P}\mathfrak{A}^{-1}_{2m+1}\mathfrak{A}_{2m-1}\mathfrak{Q}\mathfrak{A}^{-1}_{2m} \tag{4.1}$$

where, for instance,

$$\mathfrak{A}_{2m} = \begin{pmatrix} 1 & 0 \\ a_{2m} & 1 \end{pmatrix}.$$

This implies the discrete-time Zakharov–Shabat system

$$\widetilde{L}_j M_j = M_{j+1} L_j \tag{4.2}$$

where

$$L_j = \begin{pmatrix} q & 1 \\ k^2 - q^2 & 0 \end{pmatrix} \mathfrak{A}^{-1}_{2j+2}\mathfrak{A}_{2j}\mathfrak{P}\mathfrak{A}^{-1}_{2j+1}\mathfrak{A}_{2j-1}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \tag{4.3}$$

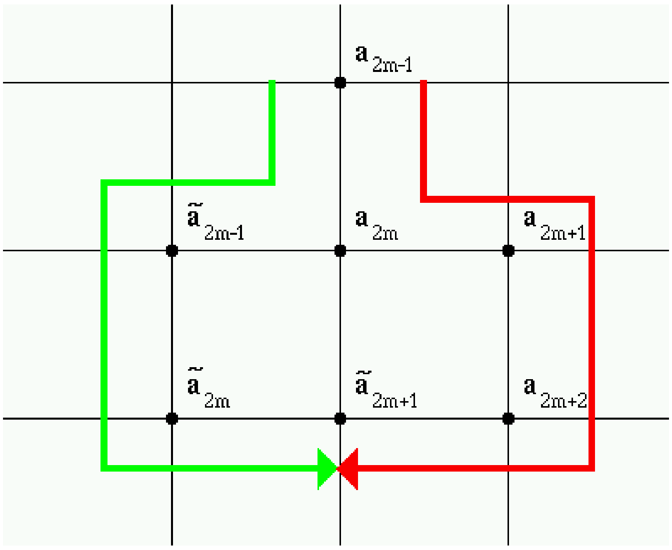

Figure 3. The Lax matrices for the mappings are derived by a consideration of the compatibility of the two paths shown.

and

$$M_j = \begin{pmatrix} q & 1 \\ k^2-q^2 & 0 \end{pmatrix} \widetilde{\mathfrak{A}}_{2j}^{-1}\mathfrak{A}_{2j} \begin{pmatrix} -p & 1 \\ k^2 & -p \end{pmatrix} \widetilde{\mathfrak{A}}_{2j-1}^{-1}\mathfrak{A}_{2j-1} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}. \tag{4.4}$$

Using the definitions (3.3), the lattice KdV (2.1), and defining

$$\lambda := k^2 - q^2$$

gives

$$L_j = V_{2j}V_{2j-1} \qquad V_{2j} = \begin{pmatrix} v_{2j} & 1 \\ \lambda & 0 \end{pmatrix} \qquad V_{2j-1} = \begin{pmatrix} v_{2j-1} & 1 \\ \lambda + a & 0 \end{pmatrix} \tag{4.5}$$

and

$$M_j = W_{2j}W_{2j-1} \qquad W_{2j} = \begin{pmatrix} a\tilde{v}_{2j-1}^{-1} & 1 \\ \lambda & 0 \end{pmatrix} \qquad W_{2j-1} = \begin{pmatrix} av_{2j-2}^{-1} & 1 \\ \lambda + a & 0 \end{pmatrix}. \tag{4.6}$$

Consider now the end points. For M_1 let $v_0 \to \infty$, hence

$$M_1 = \begin{pmatrix} a\tilde{v}_1^{-1} & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda + a & 0 \end{pmatrix} = \begin{pmatrix} \lambda + a & a\tilde{v}_1^{-1} \\ 0 & \lambda \end{pmatrix}. \tag{4.7}$$

For the last point in the even staircase special consideration must be given to M_M, let $\tilde{v}_{N-1} \to -\infty$, hence

$$M_M = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} av_{N-2}^{-1} & 1 \\ \lambda + a & 0 \end{pmatrix} = \begin{pmatrix} \lambda + a & 0 \\ \lambda a v_{N-2}^{-1} & \lambda \end{pmatrix}. \tag{4.8}$$

It is easily verified, by a straightforward calculation, that the mapping that follows from an initial values staircase with an even number of points (3.4) follows from the Zakharov–Shabat equation (4.2), along with equations (4.5), (4.6), (4.7) and (4.8). The L-matrices (4.5) are the same as those of the mappings which arise from the lattice KdV through a periodic

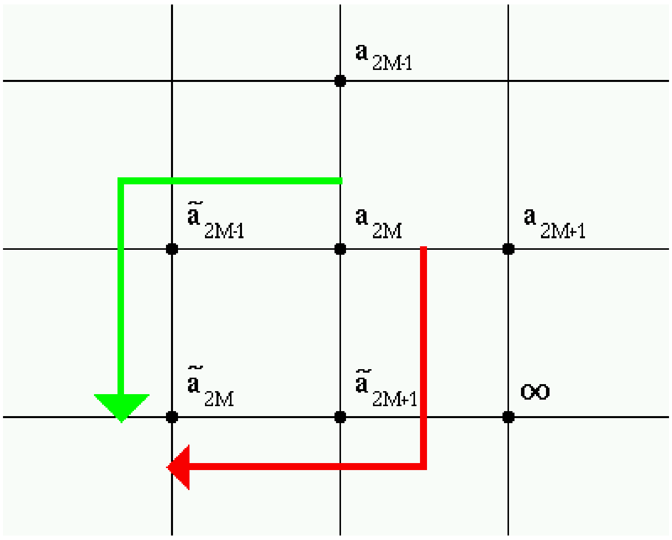

Figure 4. Additional Lax matrices for the mappings arising from an odd initial value staircase are derived by a consideration of the compatibility of the two paths shown.

initial value problem, see [22]. It should also be noted that no commuting is required in the derivation of the Lax matrices, or in the use of the compatibility equation (4.2) to derive the mapping.

For the mappings that follow from an initial value staircase with an odd number of points, extra consideration must be given to the last part of the staircase. Consider the compatibility of the paths shown in figure 4. The anticlockwise path $a_{2M} \to \tilde{a}_{2M-1} \to \tilde{a}_{2M}$ is effected by

$$\widetilde{\mathfrak{M}}_{2M-1}\widetilde{\mathfrak{L}}^{-1}_{2M-1}.$$

The clockwise path $a_{2M} \to \tilde{a}_{2M+1} \to \tilde{a}_{2M}$ is effected by

$$\widetilde{\mathfrak{L}}^{-1}_{2M}\mathfrak{M}_{2M}.$$

Equating the anticlockwise and clockwise paths gives

$$\widetilde{\mathfrak{A}}_{2M-1}\mathfrak{Q}\widetilde{\mathfrak{A}}^{-1}_{2M}\mathfrak{A}_{2M}\mathfrak{P}^{-1}\widetilde{\mathfrak{A}}^{-1}_{2M-1} = \widetilde{\mathfrak{A}}_{2M+1}\mathfrak{P}^{-1}\widetilde{\mathfrak{A}}^{-1}_{2M}\mathfrak{A}_{2M}\mathfrak{Q}\mathfrak{A}^{-1}_{2M+1}. \tag{4.9}$$

This implies the discrete-time Zakharov–Shabat style system

$$\widetilde{V}_{2M-1}M_M = N_M V_{2M-1} \tag{4.10}$$

where using the definitions (3.3), the lattice KdV (2.1), and letting $\tilde{v}_{N-1} \to -\infty$,

$$N_M = \begin{pmatrix} \lambda + a & 0 \\ (\lambda + a)a\tilde{v}^{-1}_{2M-1} & \lambda \end{pmatrix}. \tag{4.11}$$

It is easily verified, by a straightforward calculation, that the mapping that follows from an initial values staircase with an odd number of points (3.6) follows from the Zakharov–Shabat equations (4.2) and (4.10), along with equations (4.5), (4.6), (4.7), and (4.11). Again, it should also be noted that no commuting is required in the derivation of the Lax matrices, or in the use of the compatibility equations (4.2) and (4.10) to derive the mapping.

5. Invariants

The derivation of the invariants for the discrete-time evolution given by the mappings presented in sections 3.1 and 3.2 will be given in this section, using the Lax matrices derived in section 4. As before, we first consider the mapping that follows from an initial value staircase with an even number of points (3.4), before turning to the mapping that follows from one with an odd number.

Consider the transfer matrix, $T(\lambda)$, for the even case (which is obtained by gluing the elementary translation matrices L_j along a line connecting the sites 1 and l_E, where l_E is the number of degrees-of-freedom)

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} := \overleftarrow{\prod_{n=1}^{l_E}} L_n(\lambda). \tag{5.1}$$

Now consider the time update of this transfer matrix,

$$\widetilde{T}(\lambda) = \overleftarrow{\prod_{n=1}^{l_E}} \widetilde{L}_n(\lambda) = \overleftarrow{\prod_{n=1}^{l_E}} M_{n+1} L_n(\lambda) M_n^{-1} = M_{l_E+1} T(\lambda) M_1^{-1}. \tag{5.2}$$

Rewriting so that all of the time-updated variables occur on the left-hand side of the equation and all of the non-updated variables occur on the right-hand side of the equation gives

$$\widetilde{T}(\lambda) M_1 = M_{l_E+1} T(\lambda)$$

and calculating explicitly reveals that

$$\widetilde{T}(\lambda) M_1 = \begin{pmatrix} (\lambda + a)\widetilde{A} & \widetilde{A} a \tilde{v}_1^{-1} + \lambda \widetilde{B} \\ (\lambda + a)\widetilde{C} & \widetilde{C} a \tilde{v}_1^{-1} + \lambda \widetilde{D} \end{pmatrix} \tag{5.3}$$

$$M_{l_E+1} T(\lambda) = \begin{pmatrix} (\lambda + a) A & (\lambda + a) B \\ \lambda a v_{2l_E}^{-1} A + \lambda C & \lambda a v_{2l_E}^{-1} B + \lambda D \end{pmatrix}. \tag{5.4}$$

Therefore $\widetilde{A}(\lambda) = A(\lambda)$, i.e., $A(\lambda)$ is invariant under the discrete-time evolution given by the mapping (3.4).

The Lax matrices (4.5) may be written in the form

$$L_j = \Upsilon_j + \lambda \Lambda_j$$

where

$$\Upsilon_j = \begin{pmatrix} v_{2j} v_{2j-1} + a & v_{2j} \\ 0 & 0 \end{pmatrix} \qquad \Lambda_j = \begin{pmatrix} 1 & 0 \\ v_{2j-1} & 1 \end{pmatrix}.$$

The matrices Υ_j and Λ_j have the properties that, for $m > n$,

$$\overleftarrow{\prod_{j=n}^{m}} \Upsilon_j = \left(\overleftarrow{\prod_{j=n+1}^{m}} (v_{2j} v_{2j-1} + a) \right) \Upsilon_n$$

$$\overleftarrow{\prod_{j=n}^{m}} \Lambda_j = \begin{pmatrix} 1 & 0 \\ \sum_{j=n}^{m} v_{2j-1} & 1 \end{pmatrix}.$$

Hence the transfer matrix (5.1) has the gradation

$$T(\lambda) = \begin{pmatrix} \lambda^{l_E} + \lambda^{l_E-1} A_{l_E-1} + \cdots + A_0 & \lambda^{l_E-1} B_{l_E-1} + \lambda^{l_E-2} B_{l_E-2} + \cdots + B_0 \\ \lambda^{l_E} C_{l_E} + \lambda^{l_E-1} C_{l_E-1} + \cdots + \lambda C_1 & \lambda^{l_E} + \lambda^{l_E-1} D_{l_E-1} + \cdots + \lambda D_1 \end{pmatrix}. \tag{5.5}$$

More specifically

$$A_0 = \prod_{j=1}^{\overleftarrow{l_E}} (v_{2j}v_{2j-1} + a)$$

$$A_1 = \sum_{n=1}^{l_E} \left(\prod_{\substack{j=1\\ j\neq n}}^{\overleftarrow{l_E}} (v_{2j}v_{2j-1} + a) + \prod_{j=n+2}^{\overleftarrow{l_E}} (v_{2j}v_{2j-1} + a) v_{2n+2}v_{2n-1} \prod_{j=1}^{\overleftarrow{n-1}} (v_{2j}v_{2j-1} + a) \right) \tag{5.6}$$

$$\vdots$$

$$A_{l_E-1} = \sum_{n=1}^{l_E} \left(v_{2n} \left(\sum_{j=1}^{n} v_{2j-1} \right) + a \right)$$

and

$$C_1 = v_{2l_E-1} \prod_{j=1}^{\overleftarrow{l_E-1}} (v_{2j}v_{2j-1} + a)$$

$$\vdots$$

$$C_{l_E-1} = \sum_{n=1}^{l_E-1} \left(\left(\sum_{i=n+1}^{l_E} v_{2i-1} \right) \left(v_{2n} \left(\sum_{j=1}^{n} v_{2j-1} \right) + a \right) \right) \tag{5.7}$$

$$C_{l_E} = \sum_{j=1}^{l_E} v_{2j-1}.$$

Hence, the coefficients of $\lambda^0, \lambda, \lambda^2, \ldots, \lambda^{l_E-1}$ in $A(\lambda)$ give the l_E nontrivial invariants of the discrete-time evolution. Their commutativity (in the quantum case) will be proven in section 7.

Consider now the transfer matrix for the odd case (which has $l_O = M - 1$ degrees-of-freedom, where $2M + 1$ is the number of points of the initial value staircase on the lattice). For later convenience the transfer matrix and its entries, for the mappings which originate from an initial value staircase with an odd number of points, will be given in the San Serif typeface,

$$\mathsf{T}(\lambda) = \begin{pmatrix} \mathsf{A}(\lambda) & \mathsf{B}(\lambda) \\ \mathsf{C}(\lambda) & \mathsf{D}(\lambda) \end{pmatrix} := V_{2l_O+1} \prod_{n=1}^{\overleftarrow{l_O}} L_n(\lambda). \tag{5.8}$$

Now consider the time update of this transfer matrix,

$$\begin{aligned} \widetilde{\mathsf{T}}(\lambda) &= \widetilde{V}_{2l_O+1} \prod_{n=1}^{\overleftarrow{l_O}} \widetilde{L}_n(\lambda) = N_{l_O+1} V_{2l_O+1} M_{l_O+1}^{-1} \prod_{n=1}^{\overleftarrow{l_O}} M_{n+1} L_n(\lambda) M_n^{-1} \\ &= N_{l_O+1} \mathsf{T}(\lambda) M_1^{-1}. \end{aligned} \tag{5.9}$$

As before, write

$$\widetilde{\mathsf{T}}(\lambda) M_1 = N_{l_O+1} \mathsf{T}(\lambda)$$

calculating the left-hand side explicitly is obviously the same as (5.3), the right-hand side is now

$$N_{l_O+1}\mathsf{T}(\lambda) = \begin{pmatrix} (\lambda + a)\mathsf{A} & (\lambda + a)\mathsf{B} \\ (\lambda + a)a\tilde{v}_{2l_O+1}^{-1}\mathsf{A} + \lambda\mathsf{C} & (\lambda + a)a\tilde{v}_{2l_O+1}^{-1}\mathsf{B} + \lambda\mathsf{D} \end{pmatrix}. \tag{5.10}$$

Therefore $\widetilde{\mathsf{A}}(\lambda) = \mathsf{A}(\lambda)$, i.e., $\mathsf{A}(\lambda)$ is invariant under the discrete-time evolution given by the mapping (3.6).

The transfer matrix (5.8) has the grading

$$\mathsf{T}(\lambda) = \begin{pmatrix} \lambda^{l_O}\mathsf{A}_{l_O} + \lambda^{l_O-1}\mathsf{A}_{l_O-1} + \cdots + \mathsf{A}_0 & \lambda^{l_O} + \lambda^{l_O-1}\mathsf{B}_{l_O-1} + \cdots + \mathsf{B}_0 \\ (\lambda+a)\big(\lambda^{l_O} + \lambda^{l_O-1}\mathsf{C}_{l_O-1} + \cdots + \mathsf{C}_0\big) & (\lambda+a)\big(\lambda^{l_O-1}\mathsf{D}_{l_O-1} + \cdots + \mathsf{D}_0\big) \end{pmatrix}, \tag{5.11}$$

where, more specifically,

$$\begin{aligned}
\mathsf{A}_0 &= v_{2l_O+1}\prod_{j=1}^{\overleftarrow{l_O}}(v_{2j}v_{2j-1}+a)\\
\mathsf{A}_1 &= v_{2l_O+1}\sum_{n=1}^{l_O}\left(\prod_{\substack{j=1\\ j\neq n}}^{\overleftarrow{l_O}}(v_{2j}v_{2j-1}+a) + \prod_{j=n+2}^{\overleftarrow{l_O}}(v_{2j}v_{2j-1}+a)v_{2n+2}v_{2n-1}\prod_{j=1}^{\overleftarrow{n-1}}(v_{2j}v_{2j-1}+a)\right)\\
&\quad + v_{2l_O-1}\prod_{j=1}^{\overleftarrow{l_O-1}}(v_{2j}v_{2j-1}+a)\\
&\ \vdots\\
\mathsf{A}_{l_O-1} &= v_{2l_O+1}\sum_{n=1}^{l_O}\left(v_{2n}\left(\sum_{j=1}^{n}v_{2j-1}\right)+a\right) + \sum_{n=1}^{l_O-1}\left(\left(\sum_{i=n+1}^{l_O}v_{2i-1}\right)\left(v_{2n}\left(\sum_{j=1}^{n}v_{2j-1}\right)+a\right)\right)\\
\mathsf{A}_{l_O} &= \sum_{j=1}^{l_O+1}v_{2j-1}
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
\mathsf{C}_0 &= \prod_{j=1}^{\overleftarrow{l_O}}(v_{2j}v_{2j-1}+a)\\
\mathsf{C}_1 &= \sum_{n=1}^{l_O}\left(\prod_{\substack{j=1\\ j\neq n}}^{\overleftarrow{l_O}}(v_{2j}v_{2j-1}+a) + \prod_{j=n+2}^{\overleftarrow{l_O}}(v_{2j}v_{2j-1}+a)v_{2n+2}v_{2n-1}\prod_{j=1}^{\overleftarrow{n-1}}(v_{2j}v_{2j-1}+a)\right)\\
&\ \vdots\\
\mathsf{C}_{l_O-1} &= \sum_{n=1}^{l_O}\left(v_{2n}\left(\sum_{j=1}^{n}v_{2j-1}\right)+a\right).
\end{aligned} \tag{5.13}$$

The coefficient of λ^{l_O} in $\mathsf{A}(\lambda)$ is the Casimir (3.7). The coefficients of $\lambda^0, \lambda, \lambda^2, \ldots, \lambda^{l_O-1}$ in $\mathsf{A}(\lambda)$ give the l_O nontrivial invariants of the discrete-time evolution. Their commutativity (in the quantum case) will be proven in section 7.

6. Exact solution of the discrete operator equations of motion

It is generally impossible to explicitly solve the quantum-mechanical Heisenberg equations of motion. The long-standing exception to this being the harmonic oscillator. In the papers [3–5] C M Bender and G V Dunne considered the exact solution of quantum mechanical Heisenberg equations of motion for one degree-of-freedom systems (leaving higher-dimensional considerations as an open problem). They presented a method of constructing an implicit solution (which was demonstrated for systems such as the anharmonic oscillator, $H = \frac{1}{2}p^2 + \frac{1}{4}q^4$), and for certain systems they presented explicit solutions in closed form. Indeed, in [3] they showed that for Euler Hamiltonians (i.e., Hamiltonians that are solely functions of qp, for example, $H = qp$) the operator equations of motion can always be solved explicitly in closed form.

In this section the difference equations defined on the associative algebra $\mathcal{A}$ given earlier are explicitly solved as equations of motion. These are the first examples of explicitly solved *discrete* operator equations of motion, and also the first examples of explicitly solved operator equations of motion for systems with more than one degree-of-freedom. (The quantum case is given by a particular choice of commutation relations, see section 7.) The solution of the operator equations of motion is accomplished by employing the Lax matrix structure to pass to a new set of coordinates (the $\{C_i\}$ and $\{A_i\}$ of the transfer matrix for the 'even' mappings and the $\{\mathsf{C}_i\}$ and $\{\mathsf{A}_i\}$ for the 'odd' mappings) in terms of which the evolution linearizes. After evolving linearly, the original $\{v_i\}$ variables at any time-level may be reconstructed in terms of the $\{C_i\}$ and $\{A_i\}$ for the 'even' mappings, or the $\{\mathsf{C}_i\}$ and $\{\mathsf{A}_i\}$ for the 'odd' mappings, at that particular time-level.

6.1. Linear C_i evolution. Even case

Equations (5.3) and (5.4) show that

$$(\lambda + a)\frac{\widetilde{C}}{\lambda} = a v_{2l_E}^{-1} A + C. \tag{6.1}$$

Letting $\lambda \to 0$ reveals

$$v_{2l_E}^{-1} = \widetilde{C}_1 A_0^{-1}. \tag{6.2}$$

Therefore equation (6.1) gives

$$C = (\lambda + a)\frac{\widetilde{C}}{\lambda} - a\widetilde{C}_1 A_0^{-1} A \tag{6.3}$$

and, employing the gradation (5.5), one obtains,

$$\begin{aligned}
C_1 &= \widetilde{C}_1\big(1 - aA_0^{-1}A_1\big) + a\widetilde{C}_2\\
C_2 &= -a\widetilde{C}_1 A_0^{-1} A_2 + \widetilde{C}_2 + a\widetilde{C}_3\\
C_3 &= -a\widetilde{C}_1 A_0^{-1} A_3 + \widetilde{C}_3 + a\widetilde{C}_4\\
&\vdots\\
C_{l_E-1} &= -a\widetilde{C}_1 A_0^{-1} A_{l_E-1} + \widetilde{C}_{l_E-1} + a\widetilde{C}_{l_E}\\
C_{l_E} &= -a\widetilde{C}_1 A_0^{-1} + \widetilde{C}_{l_E}.
\end{aligned} \tag{6.4}$$

This autonomous linear evolution may be rewritten in the matricial form

$$
\left(\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \ldots, \widetilde{C}_{l_E}\right) = \left(C_1, C_2, C_3, \ldots, C_{l_E}\right) \times \begin{pmatrix} 1 - aA_0^{-1}A_1 & -aA_0^{-1}A_2 & -aA_0^{-1}A_3 & \ldots & -aA_0^{-1} \\ a & 1 & 0 & & 0 \\ 0 & a & 1 & & 0 \\ 0 & 0 & a & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}^{-1}. \tag{6.5}
$$

The $l_E \times l_E$ matrix on the right of (6.5) is invariant under time-updates (as it is only a function of the invariants $\{A_i\}$). It should also be remembered that the invariants $\{A_i\}$ commute with each other. Rewriting (6.5) as

$$
\widetilde{\boldsymbol{C}} = \boldsymbol{CM} \tag{6.6}
$$

it is easily seen that at time n,

$$
\boldsymbol{C}(n) = \boldsymbol{CM}^n, \tag{6.7}
$$

giving the $\{C_i\}$ variables at time n solely in terms of $\{A_i\}$ and $\{C_i\}$ at the original time. This evolution for the specific case of $l_E = 1$ appears in section 8.

In the classical case the $l_E \times l_E$ matrix on the right of (6.5) is generically diagonalizable. This allows one to speak of an interpolating continuous-time flow. In cases above one degree-of-freedom it is not clear whether such an interpretation can be applied to the general (associative algebra) case, where the invariants are not given by numerical values, but are, rather, operators. However, it does seem possible to give this interpretation to the quantum case with one degree-of-freedom (see section 8).

6.2. Linear C_i evolution. Odd case

Equations (5.3) and (5.10) show that

$$
(\lambda + a)\widetilde{\mathsf{C}} = (\lambda + a)a\tilde{v}_{2l_O+1}^{-1}\mathsf{A} + \lambda\mathsf{C}. \tag{6.8}
$$

Setting $\lambda = 0$ reveals

$$
v_{2l_0+1}^{-1} = \widetilde{\mathsf{C}}_0\mathsf{A}_0^{-1}. \tag{6.9}
$$

Therefore equation (6.8) gives

$$
\lambda\mathsf{C} = (\lambda + a)\widetilde{\mathsf{C}} - (\lambda + a)a\widetilde{\mathsf{C}}_0\mathsf{A}_0^{-1}\mathsf{A} \tag{6.10}
$$

and, employing the gradation (5.11), one obtains,

$$
\begin{aligned}
\mathsf{C}_0 &= \widetilde{\mathsf{C}}_0\left(1 - a\mathsf{A}_0^{-1}\mathsf{A}_1\right) + a\widetilde{\mathsf{C}}_1 \\
\mathsf{C}_1 &= -a\widetilde{\mathsf{C}}_0\mathsf{A}_0^{-1}\mathsf{A}_2 + \widetilde{\mathsf{C}}_1 + a\widetilde{\mathsf{C}}_2 \\
\mathsf{C}_2 &= -a\widetilde{\mathsf{C}}_0\mathsf{A}_0^{-1}\mathsf{A}_3 + \widetilde{\mathsf{C}}_2 + a\widetilde{\mathsf{C}}_3 \\
&\vdots \\
\mathsf{C}_{l_O-1} &= -a\widetilde{\mathsf{C}}_0\mathsf{A}_0^{-1}\mathsf{A}_{l_O} + \widetilde{\mathsf{C}}_{l_O-1} + a.
\end{aligned} \tag{6.11}
$$

This autonomous linear evolution may be rewritten in the matricial form

$$\left(\widetilde{\mathsf{C}}_0, \widetilde{\mathsf{C}}_1, \ldots, \widetilde{\mathsf{C}}_{l_O-1}, 1\right) = \left(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{l_O-1}, 1\right) \times \begin{pmatrix} 1 - a\mathsf{A}_0^{-1}\mathsf{A}_1 & -a\mathsf{A}_0^{-1}\mathsf{A}_2 & \ldots & -a\mathsf{A}_0^{-1}\mathsf{A}_{l_O} & 0 \\ a & 1 & & 0 & 0 \\ 0 & a & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & a & 1 \end{pmatrix}^{-1}. \tag{6.12}$$

The $l_O \times l_O$ matrix on the right of (6.12) is invariant under time-updates (as it is only a function of the invariants $\{\mathsf{A}_i\}$). It should also be remembered that the invariants $\{\mathsf{A}_i\}$ commute with each other. Rewriting (6.12) as

$$\widetilde{\mathbf{C}} = \mathbf{CM} \tag{6.13}$$

it is easily seen that at time n,

$$\mathbf{C}(n) = \mathbf{CM}^n, \tag{6.14}$$

giving the $\{\mathsf{C}_i\}$ variables at time n solely in terms of $\{\mathsf{A}_i\}$ and $\{\mathsf{C}_i\}$ at the original time.This evolution for the specific case of $l_O = 1$ appears in section 9.

Classically the $l_O \times l_O$ matrix on the right of (6.12) is generically diagonalizable. The comments at the end of the previous section also apply here.

6.3. Reconstruction

In this section the reconstruction of the original $\{v_i\}$ variables in terms of the $\{C_i\}$ and $\{A_i\}$ variables is considered (or in terms of the $\{\mathsf{C}_i\}$ and $\{\mathsf{A}_i\}$ variables in the 'odd' case). It is expedient to construct an efficient algorithmic method for this reconstruction. In [17] reconstruction of the $\{v_i\}$ dynamical variables in terms of the Sklyanin coordinates is accomplished via a 'spatial' evolution along one period of the 'spatial' (equal time) chain. For the mappings considered in this paper, which originate from a finite, rather than a periodic, initial value problem, this is not a viable possibility. Instead, reconstruction is achieved by 'peeling away' the transfer matrix, in a manner which will be made clear below.

As the 'peeling away' of the transfer matrix proceeds, we are gradually left with transfer matrices corresponding to mappings with fewer and fewer 'degrees-of-freedom'. For this reason it is necessary to introduce an addition to the notation that has been used so far in this paper, so that the number of degrees-of-freedom of the transfer matrix and its entries may be easily seen. The addition to the notation is an additional subscript, a semicolon followed by the number of degrees of freedom, that is,

$$T_{;l_E}(\lambda) = \begin{pmatrix} A_{;l_E}(\lambda) & B_{;l_E}(\lambda) \\ C_{;l_E}(\lambda) & D_{;l_E}(\lambda) \end{pmatrix} := \prod_{n=1}^{\overleftarrow{l_E}} L_n(\lambda)$$

where, for instance, $A_{;l_E}(\lambda) = \lambda^{l_E} + \lambda^{l_E-1} A_{l_E-1;l_E} + \cdots + A_{0;l_E}$, and similarly

$$\mathsf{T}_{;l_O}(\lambda) = \begin{pmatrix} \mathsf{A}_{;l_O}(\lambda) & \mathsf{B}_{;l_O}(\lambda) \\ \mathsf{C}_{;l_O}(\lambda) & \mathsf{D}_{;l_O}(\lambda) \end{pmatrix} := V_{2l_O+1} \prod_{n=1}^{\overleftarrow{l_O}} L_n(\lambda).$$

Along with the definition of the L-matrix in terms of V-matrices, equation (4.5), it is seen that $V_{2n}^{-1}T_{;n} = \mathsf{T}_{;n-1}$ and $V_{2n-1}^{-1}\mathsf{T}_{;n-1} = T_{;n-1}$. Hence,

$$\begin{pmatrix} 0 & \lambda^{-1} \\ 1 & -\lambda^{-1}v_{2n} \end{pmatrix}\begin{pmatrix} A_{;n} & B_{;n} \\ C_{;n} & D_{;n} \end{pmatrix} = \begin{pmatrix} \lambda^{-1}C_{;n} & \lambda^{-1}D_{;n} \\ A_{;n} - \lambda^{-1}v_{2n}C_{;n} & B_{;n} - \lambda^{-1}v_{2n}D_{;n} \end{pmatrix} = \begin{pmatrix} \mathsf{A}_{;n-1} & \mathsf{B}_{;n-1} \\ \mathsf{C}_{;n-1} & \mathsf{D}_{;n-1} \end{pmatrix} \tag{6.15}$$

and, similarly,

$$\begin{pmatrix} (\lambda+a)^{-1}\mathsf{C}_{;n-1} & (\lambda+a)^{-1}\mathsf{D}_{;n-1} \\ \mathsf{A}_{;n-1} - (\lambda+a)^{-1}v_{2n-1}\mathsf{C}_{;n-1} & \mathsf{B}_{;n-1} - (\lambda+a)^{-1}v_{2n-1}\mathsf{D}_{;n-1} \end{pmatrix} = \begin{pmatrix} A_{;n-1} & B_{;n-1} \\ C_{;n-1} & D_{;n-1} \end{pmatrix}. \tag{6.16}$$

Bearing in mind the gradations given in equations (5.5) and (5.11), one obtains

$$v_{2n} = -aA_{;n}(-a)(C_{;n}(-a))^{-1} \tag{6.17}$$

and

$$v_{2n-1} = \mathsf{A}_{0;n-1}\mathsf{C}_{0;n-1}^{-1}. \tag{6.18}$$

Obviously it remains to express $\{C_{i;n}\}$ and $\{A_{i;n}\}$, and $\{\mathsf{C}_{i;n}\}$ and $\{\mathsf{A}_{i;n}\}$ in terms of $\{C_{i;l_E}\}$ and $\{A_{i;l_E}\}$ in the 'even' case and in terms of $\{\mathsf{C}_{i;l_O}\}$ and $\{\mathsf{A}_{i;l_O}\}$ in the 'odd' case.

Again noting the gradations given in equations (5.5) and (5.11), and considering the $\mathsf{A}_{;n-1}$ entry of (6.15), one obtains

$$C_{j;n} = \mathsf{A}_{j-1;n-1} \qquad 1 \leqslant j \leqslant n. \tag{6.19}$$

Considering the $\mathsf{C}_{;n-1}$ entry of (6.15) one obtains

$$\begin{aligned} &A_{0;n} - v_{2n}C_{1;n} = a\mathsf{C}_{0;n-1} \\ &A_{j-1;n} - v_{2n}C_{j;n} = \mathsf{C}_{j-2;n-1} + a\mathsf{C}_{j-1;n-1} \qquad 1 < j < n \\ &A_{n-1;n} - v_{2n}C_{n;n} = \mathsf{C}_{n-2;n-1} + a. \end{aligned} \tag{6.20}$$

On considering the $A_{;n-1}$ entry of (6.16), one obtains

$$\mathsf{C}_{j;n-1} = A_{j;n-1} \qquad 0 \leqslant j \leqslant n-2. \tag{6.21}$$

Considering the $C_{;n-1}$ entry of (6.16) one obtains

$$\begin{aligned} &\mathsf{A}_{0;n-1} - v_{2n-1}\mathsf{C}_{0;n-1} = 0 \\ &\mathsf{A}_{j;n-1} - v_{2n-1}\mathsf{C}_{j;n-1} = C_{j;n-1} \qquad 0 < j < n-1 \\ &\mathsf{A}_{n-1;n-1} - v_{2n-1} = C_{n-1;n-1}. \end{aligned} \tag{6.22}$$

The reconstruction goes in essentially the same way for the 'even' and the 'odd' mappings. Equations (6.19) and (6.20) and equations (6.21) and (6.22) are put into a matricial form. The only difference being that if we are considering an 'even' mapping with l_E degrees-of-freedom, the matricial equation is in terms of $2l_E$-dimensional vectors and $2l_E \times 2l_E$ matrices, and if we are considering an 'odd' mapping with l_O degrees-of-freedom the matricial equation is in terms of $2(l_O + 1)$-dimensional vectors and $2(l_O + 1) \times 2(l_O + 1)$ matrices.

It is easily seen that the matricial equations are

$$
\left(\begin{array}{ccccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & & \vdots & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & \vdots & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a
\end{array}\right)
\left(\begin{array}{c}
\mathsf{A}_{0;n-1} \\ \mathsf{A}_{1;n-1} \\ \vdots \\ \mathsf{A}_{n-1;n-1} \\ 0 \\ \vdots \\ 0 \\
\hline
\mathsf{C}_{0;n-1} \\ \mathsf{C}_{1;n-1} \\ \vdots \\ \mathsf{C}_{n-2;n-1} \\ 1 \\ 0 \\ \vdots
\end{array}\right)
$$

$$
= \left(\begin{array}{ccccccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & \vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n} & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n}
\end{array}\right)
\left(\begin{array}{c}
A_{0;n} \\ A_{1;n} \\ \vdots \\ A_{n-1;n} \\ 1 \\ 0 \\ \vdots \\
\hline
C_{1;n} \\ C_{2;n} \\ \vdots \\ C_{n-1;n} \\ C_{n;n} \\ 0 \\ \vdots
\end{array}\right)
\qquad (6.23)
$$

$$
\left(\begin{array}{ccccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & & \vdots & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & \vdots & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
\left(\begin{array}{c}
A_{0;n-1} \\ A_{1;n-1} \\ \vdots \\ A_{n-2;n-1} \\ 1 \\ 0 \\ \vdots \\
\hline
C_{1;n-1} \\ C_{2;n-1} \\ \vdots \\ C_{n-1;n-1} \\ 0 \\ \vdots \\ 0
\end{array}\right)
$$

$$
= \left(\begin{array}{ccccccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & \vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n-1} & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{2n-1}
\end{array}\right)
\left(\begin{array}{c}
\mathsf{A}_{0;n-1} \\ \mathsf{A}_{1;n-1} \\ \vdots \\ \mathsf{A}_{n-2;n-1} \\ \mathsf{A}_{n-1;n-1} \\ 0 \\ \vdots \\
\hline
\mathsf{C}_{0;n-1} \\ \mathsf{C}_{1;n-1} \\ \vdots \\ \mathsf{C}_{n-2;n-1} \\ 1 \\ 0 \\ \vdots
\end{array}\right).
\tag{6.24}
$$

To take the specific example of an 'even' mapping, reconstruction would proceed as follows. One immediately has v_{2l_E} in terms of $\{A_{i;l_E}\}$ and $\{C_{i;l_E}\}$ from equation (6.17). Now, from equation (6.18), v_{2l_E-1} is expressed in terms of $\mathsf{A}_{0;l_{E-1}}$ and $\mathsf{C}_{0;l_{E-1}}$. Using equation (6.23), in conjunction with v_{2l_E} in terms of $\{A_{i;l_E}\}$ and $\{C_{i;l_E}\}$, gives $\{\mathsf{A}_{i;l_{E-1}}\}$ and $\{\mathsf{C}_{i;l_{E-1}}\}$ in terms of $\{A_{i;l_E}\}$ and $\{C_{i;l_E}\}$. Therefore v_{2l_E-1} may be expressed in terms of $\{A_{i;l_E}\}$ and $\{C_{i;l_E}\}$. Equation (6.17) also gives v_{2l_E-2} in terms of $\{A_{i;l_E-1}\}$ and $\{C_{i;l_E-1}\}$. Using equation (6.24), in conjunction with v_{2l_E-1} in terms of $\{\mathsf{A}_{i;l_{E-1}}\}$ and $\{\mathsf{C}_{i;l_{E-1}}\}$, gives $\{A_{i;l_E-1}\}$ and $\{C_{i;l_E-1}\}$ in terms of $\{\mathsf{A}_{i;l_{E-1}}\}$ and $\{\mathsf{C}_{i;l_{E-1}}\}$ and, therefore (as we already have $\{\mathsf{A}_{i;l_{E-1}}\}$ and $\{\mathsf{C}_{i;l_{E-1}}\}$ in terms of $\{A_{i;l_E}\}$ and $\{C_{i;l_E}\}$), gives v_{2l_E-2} in terms of $\{A_{i;l_E}\}$ and $\{C_{i;l_E}\}$. The procedure continues until reconstruction of the original $\{v_i\}$ is completed. The 'odd' case is completely analogous.

7. Quantum mappings and the Yang–Baxter structure

The commutation relation

$$[v_j, v_{j'}] = G(\delta_{j,j'+1} - \delta_{j+1,j'}), \tag{7.1}$$

where G is an arbitrary time-independent member of the algebra $\mathcal{A}$, is preserved under an evolution by the even map (3.4) or the odd map (3.6). Specifying the commutation relation (7.1) allows us to consider the maps given previously as bona fide dynamical systems. With this commutation relation we see that the even map, (3.4), has $l_E := M - 1$ degrees-of-freedom, and the odd map, (3.6), has $l_O := M - 1$ degrees-of-freedom as ν, of equation (3.7), is now a Casimir.

Without specifying commutation relations for the dynamical variables $\{v_i\}$ it was possible to solve the mappings (as discrete equations of motion) for all (discrete) times. This result has echoes of an observation of [8] that the integrability, in the sense of higher-dimensional consistency, of partial difference equations defined around an elementary plaquette (as discussed in [27], and comprehensively in [1]), when the field variables belong to an associative algebra, is independent of the commutation relations between the field variables.

Integrability of dynamical systems is usually defined in terms of a sufficient number of commuting invariants of the evolution, which is usually proven using a Yang–Baxter structure. Without specifying commutation relations between $G \in \mathcal{A}$ and the dynamical variables it is not possible to prove commutativity of the invariants of the discrete-time evolution. In this section we consider $G = hI$, where $h = i\hbar \in i\mathbb{R}$ and I is the unit of $\mathcal{A}$ (and so commutes with all members of $\mathcal{A}$). Hence

$$[v_j, v_{j'}] = h(\delta_{j,j'+1} - \delta_{j+1,j'}). \tag{7.2}$$

This is the commutation relation specified for the quantum mappings of KdV type given in [22, 11]. With this commutation relation $\{v_{2j+1}\}_{j\in\mathbb{Z}}$ and $\{v_{2i}\}_{i\in\mathbb{Z}}$ form a complete set of complementary quantum observables.

In this section it will be shown that the quantum maps (3.4) and (3.6) with (7.2) are integrable (in that all of the invariants of a particular map commute pairwise). And, therefore, in the classical limit this gives a new family of classical integrable maps (in the LAV sense, see [33, 32, 10]). More specifically, the non-ultralocal Yang–Baxter structure of the L- and V-matrices belonging to the mappings will be presented. This Yang–Baxter structure allows a concise proof of the pairwise commutativity of the invariants of the discrete-time evolution for any number of degrees-of-freedom. (And hence, in the classical limit, it gives a concise proof of the involutivity of the invariants with respect to the Poisson structure.)

The Yang–Baxter structure for the mapping that follows from an initial value staircase with an even number of points, (3.4), is already known; as the L-matrices are the same as those in the periodic case the Yang–Baxter structure is the same. The only nontrivial commutation relations between the operators $L_n(\lambda)$ are those on the same and nearest-neighbour sites, namely as follows

$$R^+_{12} L_{n,1} L_{n,2} = L_{n,2} L_{n,1} R^-_{12} \tag{7.3}$$

$$L_{n+1,1} S^+_{12} L_{n,2} = L_{n,2} L_{n+1,1} \tag{7.4}$$

$$L_{n,1}L_{m,2} = L_{m,2}L_{n,1} \qquad |n-m| \geqslant 2 \tag{7.5}$$

where $L_n(\lambda)$ is the L operator at the nth site and $L_{n,j}$ denotes $L_n(\lambda)$ acting nontrivially only on the jth factor of the tensor product,

$$L_{n,j} := \mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\underbrace{L_n(\lambda_j)}_{j\text{th place}}\otimes\cdots\otimes\mathbf{1}.$$

The operators $R^{\pm}_{jk} := R^{\pm}_{jk}(\lambda_j,\lambda_k)$ act nontrivially only on the jth and kth factors of the tensor product. The realization of the R and S matrices is

$$\begin{aligned} R^+_{12} &= R^-_{12} - S^+_{12} + S^-_{12} \\ R^-_{12} &= \mathbf{1}\otimes\mathbf{1} + h\frac{P_{12}}{\lambda_1-\lambda_2} \\ S^+_{12} &= \mathbf{1}\otimes\mathbf{1} - \frac{h}{\lambda_2}F\otimes E \qquad S^-_{12} = S^+_{21} \end{aligned} \tag{7.6}$$

where the permutation operator P_{12} and the matrices E and F are given by

$$P_{12} = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \qquad E = \begin{pmatrix} 0&1\\0&0 \end{pmatrix} \qquad F = \begin{pmatrix} 0&0\\1&0 \end{pmatrix}. \tag{7.7}$$

The commutation relations encapsulated within (7.3) are given in appendix A. Equations (7.3), (7.4) and (7.5) lead to

$$R^+_{12}T_1T_2 = T_2T_1R^-_{12} \tag{7.8}$$

where T is as defined in (5.1). From which it follows (see appendix A) that the invariants (i.e., the coefficients of different powers of λ in $A(\lambda)$) commute.

For the odd mappings, (3.6), it is necessary to consider the Yang–Baxter structure for the V-matrices. This Yang–Baxter structure was alluded to in [22], but not given explicitly. Its explicit realization for the mappings which arise from the lattice KdV will now be presented. The only nontrivial commutation relations between the operators $V_n(\lambda)$ are those on the same and nearest-neighbour sites, namely as follows,

$$R^+_{12}(n)V_{n,1}V_{n,2} = V_{n,2}V_{n,1}R^-_{12} \tag{7.9}$$

$$V_{n+1,1}S^+_{12}(n)V_{n,2} = V_{n,2}V_{n+1,1} \tag{7.10}$$

$$V_{n,1}V_{m,2} = V_{m,2}V_{n,1} \qquad |n-m| \geqslant 2 \tag{7.11}$$

where $V_n(\lambda)$ is the V operator at the nth site and $V_{n,j}$ denotes $V_n(\lambda)$ acting nontrivially only on the jth factor of the tensor product. The realization of the R and S matrices is

$$\begin{aligned} R^+_{12}(n) &= R^-_{12} - S^+_{12}(n) + S^-_{12}(n) \\ R^-_{12} &= \mathbf{1}\otimes\mathbf{1} + h\frac{P_{12}}{\lambda_1-\lambda_2} \\ S^+_{12}(2j-1) &= \mathbf{1}\otimes\mathbf{1} - \frac{h}{\lambda_2+a}F\otimes E \\ S^+_{12}(2j) &= \mathbf{1}\otimes\mathbf{1} - \frac{h}{\lambda_2}F\otimes E \qquad S^-_{12}(n) = S^+_{21}(n) \end{aligned} \tag{7.12}$$

where $j \in \mathbb{Z}$, $n \in \mathbb{Z}$, the permutation operator P_{12} and the matrices E and F are as given in equation (7.7). It can be shown that

$$R^-_{12}S^-_{12}(n) = S^+_{12}(n)R^+_{12}(n).$$

Of course $R^+_{12}(2j) = R^+_{12}$ and $S^+_{12}(2j) = S^+_{12}$. The commutation relations encapsulated within (7.9) for odd n are given in appendix B. Equations (7.9), (7.10) and (7.11) lead to

$$R^+_{12}(2j-1)T_1T_2 = T_2T_1R^-_{12} \tag{7.13}$$

where T is as defined in (5.8). From which it follows (see appendix B) that the invariants (i.e., the coefficients of different powers of λ in $A(\lambda)$) also commute in the case of the mappings (3.6).

8. Specific example of the 'even' mapping

When the number of degrees-of-freedom $l_E = 1$, the mapping (3.4) becomes

$$\tilde{v}_1 = v_1 + a v_2^{-1} \qquad \tilde{v}_2 = v_2 - a\tilde{v}_1^{-1} \tag{8.1}$$

and the commutation relation between v_1 and v_2 reads

$$[v_1, v_2] = i\hbar. \tag{8.2}$$

In the classical limit this map was considered in [16] where the conversion to action-angle variables was performed, and the modified (or interpolating) Hamiltonian was given. However, here we shall continue to use the machinery of the present paper, and also maintain the commutation relation (8.2) (i.e., remain in a quantum context).

In the one degree-of-freedom case, equation (6.5) becomes

$$\widetilde{C}_1 = C_1\left(1 - aA_0^{-1}\right)^{-1} \tag{8.3}$$

which implies that C_1 at time $n \in \mathbb{Z}$, $C_1(n)$, is

$$C_1(n) = C_1\left(1 - aA_0^{-1}\right)^{-n}. \tag{8.4}$$

To complete the picture, we want to express v_1 and v_2 at any time in terms of v_1 and v_2 at the original time. The transfer matrix in the one degree-of-freedom case is simply the first Lax matrix,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} := \begin{pmatrix} \lambda + v_2v_1 + a & v_2 \\ \lambda v_1 & \lambda \end{pmatrix} \tag{8.5}$$

from which we see that the invariant $A_1 = v_2v_1 + a$, and $C_1 = v_1$. Therefore

$$v_1(n) = v_1(1 + a(v_2v_1)^{-1})^n. \tag{8.6}$$

Now, as v_2v_1 is invariant under time-updates, $v_2(n)$ follows immediately. However, continuing to use the machinery of the previous sections we see that equation (6.17) gives

$$\begin{aligned} v_2 &= -aA(-a)(C(-a))^{-1} \\ &= (A_0 - a)(C_1)^{-1}. \end{aligned} \tag{8.7}$$

Hence, along with equation (8.4) it is seen that

$$\begin{aligned} v_2(n) &= (A_0 - a)\left(1 - aA_0^{-1}\right)^n (C_1)^{-1} \\ &= (v_2v_1(v_2v_1 + a)^{-1})^n v_2. \end{aligned} \tag{8.8}$$

Therefore the discrete operator equations of motion (8.1) have been solved exactly for all integer times n. One may also consider formulae (8.6) and (8.8) as interpolating between the integer time-steps.

Hence it is seen that a function of the invariant v_2v_1 is an interpolating (or modified) Hamiltonian of the quantum mapping. As a quantum mechanical Hamiltonian $H = v_2v_1$ was considered by Berry and Keating in [6] (see also [7]). Their purpose was to present evidence that the eigenvalues (or energy levels), E_n, of the formally Hermitian quantum operator $H = v_1 v_2 - \mathrm{i}\frac{\hbar}{2}$ are the imaginary part of the complex zeros of Riemann's zeta function, i.e., that

$$\zeta\left(\tfrac{1}{2} + \mathrm{i}E_n\right) = 0.$$

The semiclassical expression derived for the statistics of E_n gave tantalizing insights into a possible connection with the zeta function, $\zeta(s)$. However, there are, as is freely admitted in the work, many difficulties and problems that are not merely technical, such as the space on which $v_1 v_2$ acts is not known (different possibilities are discussed for, somehow, 'sewing up' the phase space), and there are many analytical possibilities, for instance different boundary conditions, etc (which are considered, but it is basically concluded that there is no clear route to take).

Many of the same problems considered in [6] (such as the determination of what the correct space is for $v_1 v_2$ to act on) will have to be addressed in a more rigorous consideration of the quantum mappings considered in this paper.

9. Specific example of the 'odd' mappings

When the number of degrees-of-freedom $l_O = 1$, the mapping (3.6) becomes

$$\tilde{v}_1 = v_1 + a v_2^{-1} \qquad \tilde{v}_3 = v_3 - a v_2^{-1} \qquad \tilde{v}_2 = v_2 - a\tilde{v}_1^{-1} + a\tilde{v}_3^{-1}. \tag{9.1}$$

The Casimir of this algebra (3.7), with the commutation relations (7.2), is, in the one degree-of-freedom case,

$$\nu = \mathsf{A}_{1;1} = v_1 + v_3. \tag{9.2}$$

For convenience, v_3 will be left in some of the equations that follow; however, it should be born in mind that the meaning of v_3 is, actually, $\nu - v_1$.

With one degree-of-freedom equation (6.12) becomes

$$(\widetilde{\mathsf{C}}_0, 1) = (\mathsf{C}_0, 1)\begin{pmatrix} 1 - a\mathsf{A}_0^{-1}\mathsf{A}_1 & 0 \\ a & 1 \end{pmatrix}^{-1} = (\mathsf{C}_0, 1)\begin{pmatrix} \mathsf{A}_0\mathsf{A}(-a)^{-1} & 0 \\ -a\mathsf{A}_0\mathsf{A}(-a)^{-1} & 1 \end{pmatrix}. \tag{9.3}$$

which implies that C_0 at time $n \in \mathbb{Z}$, $\mathsf{C}_0(n)$, is

$$\mathsf{C}_0(n) = \mathsf{C}_0(\mathsf{A}_0\mathsf{A}(-a)^{-1})^n - a\sum_{j=1}^{n}(\mathsf{A}_0\mathsf{A}(-a)^{-1})^j. \tag{9.4}$$

To complete the picture, we want to express $v_1 = \mathsf{A}_1 - v_3$ and v_2 at any time-level in terms of v_1 and v_2 at the original time-level.

From equation (6.18),

$$v_3(n) = \mathsf{A}_0\mathsf{C}_0(n)^{-1} \tag{9.5}$$

equation (6.17) gives

$$v_2 = -aA_{;1}(-a)(C_{;1}(-a))^{-1} \tag{9.6}$$

and equation (6.24) reads

$$\begin{pmatrix} A_{0;1} \\ 1 \\ 0 \\ C_{1;1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -v_3 & 0 \\ 0 & 1 & 0 & -v_3 \end{pmatrix}\begin{pmatrix} \mathsf{A}_0 \\ \mathsf{A}_1 \\ \mathsf{C}_0 \\ 1 \end{pmatrix}. \tag{9.7}$$

Hence

$$A_{;1}(-a) = \mathsf{C}_0 - a \qquad C_{;1}(-a) = -a(\mathsf{A}_1 - v_3).$$

Hence

$$v_2(n) = (\mathsf{C}_0(n) - a)(\mathsf{A}_1 - \mathsf{A}_0\mathsf{C}_0(n)^{-1})^{-1}. \tag{9.8}$$

Therefore, in conjunction with (9.4), equations (9.5) and (9.8) give $v_3(n)$ (and, so, $v_1(n)$) and $v_2(n)$ in terms of A_0, $\mathsf{A}_1 = \nu$ and C_0 at the initial time.

10. Summary

A new type of initial boundary condition for lattice equations that relate values of a field variable around a quadrilateral was proposed. It was applied to the lattice KdV equation, which resulted in two new families of integrable mappings. The solution of these mappings (discrete equations of motion) can be found over any discrete time interval, even in the quantum case. These mappings are, hence, examples of multi-dimensional operator equations of motion that can be solved for all time (in the spirit of the work [3, 4] on the one degree-of-freedom case).

The field variables of the lattice KdV took values in an associative algebra from the start. This puts the present work in the context of other recent work in integrable systems such as [19, 28] where the structure of integrable PDEs was extended to the domain of associative algebras, [2] where Painlev é equations were defined on an associative algebra, and [8] where the higher-dimensional consistency (consistency around a cube) property was investigated for integrable partial difference equations defined on an associative algebra. One may speculate that developments in this area may lead towards new approaches to the quantization process.

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Appendix A. RTT commutation relations

Consider a matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

In this section we give the commutation relations for entries of a matrix of this form, which obeys the Yang–Baxter relation

$$R^+_{12} T_1 T_2 = T_2 T_1 R^-_{12}$$

where the R-matrices are the lattice KdV realization, as given in equation (7.6).

Let us use the convention that if we work with different values of the spectral parameter $\lambda_1, \lambda_2, \ldots,$ the corresponding values of the operators $A(\lambda_i), \ldots, D(\lambda_i)$ will be denoted by $A_i, \ldots, D_i$, $(i = 1, 2, \ldots)$. Using this shorthand notation we have the following list of commutation relations for the entries of $T(\lambda)$,

$$\hat{\lambda}_{12} A_1 A_2 = \hat{\lambda}_{12} A_2 A_1$$
$$\hat{\lambda}_{12} A_1 B_2 = \lambda_{12} B_2 A_1 + h A_2 B_1$$

$$\hat{\lambda}_{12} B_1 A_2 = h B_2 A_1 + \lambda_{12} A_2 B_1$$
$$\hat{\lambda}_{12} B_1 B_2 = \hat{\lambda}_{12} B_2 B_1$$
$$\lambda_{12} A_1 C_2 + h \frac{\lambda_2}{\lambda_1} C_1 A_2 = \hat{\lambda}_{12} C_2 A_1$$
$$\lambda_{12} A_1 D_2 + h \frac{\lambda_2}{\lambda_1} C_1 B_2 = \lambda_{12} D_2 A_1 + h C_2 B_1$$
$$\lambda_{12} B_1 C_2 + h \frac{\lambda_2}{\lambda_1} D_1 A_2 = h D_2 A_1 + \lambda_{12} C_2 B_1$$
$$\lambda_{12} B_1 D_2 + h \frac{\lambda_2}{\lambda_1} D_1 B_2 = \hat{\lambda}_{12} D_2 B_1$$
$$h \frac{\lambda_1}{\lambda_2} A_1 C_2 + \lambda_{12} C_1 A_2 = \hat{\lambda}_{12} A_2 C_1$$
$$h \frac{\lambda_1}{\lambda_2} A_1 D_2 + \lambda_{12} C_1 B_2 = \lambda_{12} B_2 C_1 + h A_2 D_1$$
$$h \frac{\lambda_1}{\lambda_2} B_1 C_2 + \lambda_{12} D_1 A_2 = h B_2 C_1 + \lambda_{12} A_2 D_1$$
$$h \frac{\lambda_1}{\lambda_2} B_1 D_2 + \lambda_{12} D_1 B_2 = \hat{\lambda}_{12} B_2 D_1$$
$$\hat{\lambda}_{12} C_1 C_2 = \hat{\lambda}_{12} C_2 C_1$$
$$\hat{\lambda}_{12} C_1 D_2 = \lambda_{12} D_2 C_1 + h C_2 D_1$$
$$\hat{\lambda}_{12} D_1 C_2 = h D_2 C_1 + \lambda_{12} C_2 D_1$$
$$\hat{\lambda}_{12} D_1 D_2 = \hat{\lambda}_{12} D_2 D_1,$$

where we have also introduced, for these appendices only, the notation $\hat{\lambda}_{ij} \equiv \lambda_i - \lambda_j + h$, $\lambda_{ij} \equiv \lambda_i - \lambda_j$.

Appendix B. $R(2j-1)TT$ commutation relations

Consider a matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

In this section we give the commutation relations for entries of a matrix of this form, which obeys the Yang–Baxter relation

$$R_{12}^{+}(2j-1) T_1 T_2 = T_2 T_1 R_{12}^{-}$$

where the R-matrices are the lattice KdV realization, as given in equation (7.12).

Let us use the convention that if we work with different values of the spectral parameter $\lambda_1, \lambda_2, \ldots$, the corresponding values of the operators $A(\lambda_i), \ldots, D(\lambda_i)$ will be denoted by $A_i, \ldots, D_i$, $(i = 1, 2, \ldots)$. Using this shorthand notation we have the following list of commutation relations for the entries of $T(\lambda)$,

$$\hat{\lambda}_{12} A_1 A_2 = \hat{\lambda}_{12} A_2 A_1$$
$$\hat{\lambda}_{12} A_1 B_2 = \lambda_{12} B_2 A_1 + h A_2 B_1$$
$$\hat{\lambda}_{12} B_1 A_2 = h B_2 A_1 + \lambda_{12} A_2 B_1$$
$$\hat{\lambda}_{12} B_1 B_2 = \hat{\lambda}_{12} B_2 B_1$$

$$\lambda_{12}A_1C_2 + h\frac{\lambda_2 + a}{\lambda_1 + a}C_1A_2 = \hat{\lambda}_{12}C_2A_1$$
$$\lambda_{12}A_1D_2 + h\frac{\lambda_2 + a}{\lambda_1 + a}C_1B_2 = \lambda_{12}D_2A_1 + hC_2B_1$$
$$\lambda_{12}B_1C_2 + h\frac{\lambda_2 + a}{\lambda_1 + a}D_1A_2 = hD_2A_1 + \lambda_{12}C_2B_1$$
$$\lambda_{12}B_1D_2 + h\frac{\lambda_2 + a}{\lambda_1 + a}D_1B_2 = \hat{\lambda}_{12}D_2B_1$$
$$h\frac{\lambda_1 + a}{\lambda_2 + a}A_1C_2 + \lambda_{12}C_1A_2 = \hat{\lambda}_{12}A_2C_1$$
$$h\frac{\lambda_1 + a}{\lambda_2 + a}A_1D_2 + \lambda_{12}C_1B_2 = \lambda_{12}B_2C_1 + hA_2D_1$$
$$h\frac{\lambda_1 + a}{\lambda_2 + a}B_1C_2 + \lambda_{12}D_1A_2 = hB_2C_1 + \lambda_{12}A_2D_1$$
$$h\frac{\lambda_1 + a}{\lambda_2 + a}B_1D_2 + \lambda_{12}D_1B_2 = \hat{\lambda}_{12}B_2D_1$$
$$\hat{\lambda}_{12}C_1C_2 = \hat{\lambda}_{12}C_2C_1$$
$$\hat{\lambda}_{12}C_1D_2 = \lambda_{12}D_2C_1 + hC_2D_1$$
$$\hat{\lambda}_{12}D_1C_2 = hD_2C_1 + \lambda_{12}C_2D_1$$
$$\hat{\lambda}_{12}D_1D_2 = \hat{\lambda}_{12}D_2D_1,$$

where we have also introduced, for these appendices only, the notation $\hat{\lambda}_{ij} \equiv \lambda_i - \lambda_j + h$, $\lambda_{ij} \equiv \lambda_i - \lambda_j$.

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